**Weak Field Limit**

Now let’s explore a few special cases of Einstein’s equations. The first obvious test of his equations would be to verify that they reduce to Newton’s laws in the weak field limit. So this is where we’ll start:



We’ll assume that the metric tensor gαβ is very nearly flat since we’re in the weak field limit. So we’ll say:



where the hαβ elements are all small << 1 (so we’re basically setting up a perturbation theory problem). Note that what we’re really saying is that we can find some coordinate system where this is so, because the metric matrix elements are coordinate system dependent. Note that hαβ forms the components of a tensor against the class of Lorentz coordinate transformations. This is because:



since the Lorentz transformation preserves the η metric, by design. And so we see that the metric correction, , behaves as a tensor field against the SR class of coordinate transformations. So this is our first clue that hαβ will more or less contain the components of the gravitational field. Harping on this a bit longer, we can define the mixed components of via:



(or can we say that the exact equations would have η replaced by g, but then when we go to first order, we can replace g with η?) Before diving right into Einstein’s equations, let’s also point out that we have ‘gauge’ freedom in choosing our coordinate system (basically, we can choose Cartesian, Cylindrical, etc., coordinates, as well as having temporal d.o.f.), and this can affect what hαβ looks like. For instance, consider the following change of coordinates:



where ξα is some small change in coordinates. Then our metric would change to:



So we see that there is a first order correction to g, such that h´αβ → hαβ + ξα,β + ξβ,α. But if these last terms are small, then it will still be a ‘valid’ change of coordinates because the correction to the flat metric, , will still be small. Anyway, calculating Rαβδγ from this we have,



while,



which makes, keeping terms only to first order in h:



and so,



The Ricci tensor is:



where we have defined *h* as the trace of hαβ, i.e., hαα. And the Ricci scalar is:



and so the LHS of Einstein’s equation is as follows. Note we’ll make a few seemingly arbitrary manipulations towards a simplification:



where we define,



which is called the trace reverse of . This because:



We will want to establish that it is its own inverse for later use. So observe that:



Now let’s keep in mind that we still have ‘gauge’ freedom to vary our coordinate system. Let’s do this, with a mind to eliminate the terms and such. Let’s consider this gauge freedom by introducing a new coordinate system:



This induces a change in , as aforementioned:



And now we would like to choose the ξα(*xβ*)functions to make . Can this be done? Well we would need,



So this would be our gauge condition:



which when satisfied, would make the offending term 0. So let’s suppose that we have changed coordinates to this system. Then the LHS of our Einstein equation simplifies to:



and equating to the RHS we get:



which are the linearized approximation to Einstein’s equations. Now let’s assume that matter source is moving at slow velocities so that the 00 term dominates (or that the matter is stationary so T00 is all there is). So we’ll just focus on that one. For dust, we have:



What is 2? Well, the matter is stationary, by assumption, so the spatial components of the velocity are zero. But the magnitude of the four velocity is -c2, so the temporal component is not. We can work out the temporal component therefore to first order.



So we have:



And we can write:



So this illustrates that disturbances in the metric ripple away at the speed of light – our first elementary result with gravitational waves! But if sources are static, or nearly so, then so will be the metric and the time-part of our equation will be negligible. So now we have:



We have solved this type of equation before, in the context of Newton’s original theory of gravity,



Appropriating this result, we see our solution is:



where φ is the Newtonian gravitational potential associated with that mass density. And from this we can get the elements of our metric *h*αβ. We have:



and so our entire metric is:



which makes our line element:



I’ll make a quick comment about the metric. So we saw that the gravity waves which propagate outwards from some source are ripples in the space-time metric. At the same time, we can see that the space-time metric kind of just the gravitational potential. This latter fact helps us see why when space contracts/expands, not everything contracts/expands with it. Because in a sense it is still just a ‘force’, and one among many. If the electric force binding two atoms together into a molecule is stronger than the gravitational potential comprising the metric, then ripples in the metric won’t be strong enough to pull apart the atoms, say.

**Interpretation of the time coordinate and radial coordinate**

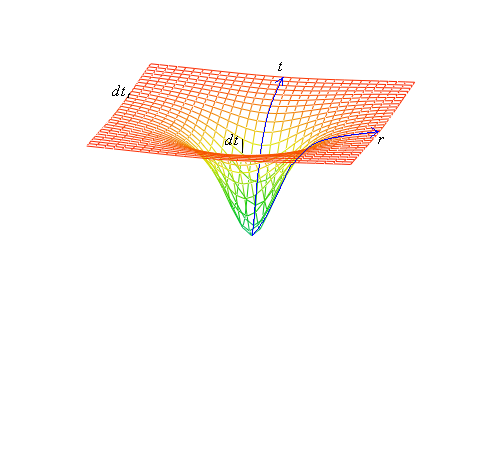
So the time-like coordinate does not represent actual time, and neither does the radial coordinate represent radial distance. Consider a stationary observer out at infinity. We’ll note that as r → ∞, g00 → -1. This tells us that physical time between events A and B for an observer relates to the coordinate time between those events according to [where the implicit xα(s) is the path of our stationary observer]:



and so basically, the physical time for an infinitely far removed person is the same as book-keeping time. We’ll also observe time dilation. Consider an event that begins and ends at book-keeping time tA,B respectively. What is the physical time interval as measured by someone at position r? This would be:



So the same event will seem to take less and less time, the further you go into the well (φ is negative by the way). We can think of this as being because the time coordinate is getting stretched out as we go into the well, kind of illustrated below:



Radial distances would be calculated via:



And we’ll note that for rA say somewhat close to the mass and rB far far away, this will just about be rB – rA because the integrand will be approximately 1 for most of the integral. So basically, far from the star, r becomes the physical radius just as t becomes the physical time.